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Lie Group Analysis of Second-Order Non-Linear Neutral Delay Differential Equations

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ABSTRACT

Lie group analysis is applied to second order neutral delay differential equations (NDDEs) to study the properties of the solution by the classification scheme. NDDE is a delay differential equation which contains the derivatives of the unknown function both with and without delays. It turns out that in many cases where retarded delay differential equation (RDDE) fail to model a problem, NDDE provides a solution. This paper extends the classification of second order non-linear RDDE to solvable Lie algebra to that for second order non-linear NDDE. In this classification the second order extension of the general infinitesimal generator acting on second order neutral delay is used to determine the determining equations. Then the resulting equations are solved, and the solvable Lie algebra is obtained, satisfying the inclusion property. Finally, one-parameter Lie groups which are corresponding to NDDEs are determined. This approach provides a theoretical background for constructing invariant solutions.

Keywords: Neutral delay differentia equation, Lie group analysis, Lie group, Lie algebra, one-parameter Lie group.

1. Introduction

Delay differential equation (DDE) was initially introduced in the 18th century by Laplace and Condorcet (see Gorecki et al., 1989). It arises when ordinary differential equations (ODEs) fail to explain some natural phenomena. Then DDEs have been successfully used in the mathematical formulation of real life phenomena in a wide variety of applications, especially in science and engineering such as population dynamics, infectious disease, control problems, secure communication, traffic control, electrodynamics and economics (Bellen and Zennaro, 2003; Batzel and Tran, 2000; Nagy et al., 2001). In contrast with ordinary differential equations (ODEs) where the unknown function and its derivatives are evaluated at the same instant, in a DDE the evolution of the system at a certain time, depends on the state of the system at an earlier time. The delay however adds extra complexities and generally DDEs are difficult to solve. When there is no direct way to solve it, we try to arrive at suitable solution by analyzing the properties of DDEs. The best way to study the properties of the solution of delay differential equation is by Lie group analysis.

Lie group analysis was introduced by Sophus Lie (see Oliveri, 2010) it is considered to be an effective method for studying the properties of differential equation (DE). Lie developed theories on continuous groups which are called Lie groups. Since then, Lie group analysis has been widely exploited (Bluman and Kumei, 1989; Hill, 1982; Olver, 1993). Tanthanuch and Meleshko, 2004, defined an admitted Lie group for functional differential equation (FDDE) which helped Pue-on, 2009, to introduce group classification for specific cases of second order delay differential equation. Most researchers deal with Lie group analysis of DEs by making a change in space variables, but DDEs do not possess equivalent transformations to change the dependent and independent variables. Because of this, some researchers (Pue-on, 2009) failed to classify DDEs as Lie algebras.

Recently, Muhsen and Maan, 2014a, introduced a classification of second order linear delay differential equation to solvable Lie algebra without changing the space variables. Then they extend the classification to second order non-linear RDDEs to solvable Lie algebra (Muhsen and Maan, 2014b). This paper extends the classification method to second order non-linear neutral delay differential equation to solvable Lie algebra and obtains the one-parameter Lie group of the corresponding NDDEs. This result is useful to study the properties of many natural phenomena which are described by non-linear NDDEs.

The content of the present paper is as follows. Section 2 gives the principal details about Lie algebra and delay differential equations. The classification of second order non-linear neutral delay differential equations to solvable Lie algebra with main results are described in Section 3. Section

4 concludes with comments on the robustness and versatility of our approach.

2. Preliminaries

This paper proposes a classification of second order non-linear neutral delay differential equations to solvable Lie algebra. We first give some information on Lie algebra, and delay differential equations.

Definition 2.1 (Andreas, 2009): A Lie algebra L is an n-dimensional solvable algebra if there exist a sequence that yields,

$$L_1 \subset L_2 \subset ... \subset L_n = L$$

Here L_k is called k-dimensional Lie algebra and L_{k-1} is an ideal of L_k $k = 1,2,...,n$ in which two dimensional Lie algebra are solvable.

Definition 2.2 (Bluman and Kumei, 1989): Let $Q_i = \zeta_s \frac{\partial}{\partial x_s}$ and $Q_j = \eta_s \frac{\partial}{\partial x_s}$, $i, j = 1,...,r$, and $s = 1,...,n$ be two infinitesimal generator. The commutator $[Q_i, Q_j]$ of Q_i and Q_j is the first order operator

$$[Q_i, Q_j] = Q_i Q_j - Q_j Q_i = \sum_s^n \sum_m^n (\zeta_m \frac{\partial \eta_s}{\partial x_m} - \eta_m \frac{\partial \zeta_s}{\partial x_m}) \frac{\partial}{\partial x_s}.$$

Definition 2.3 (Humi and Miller, 1988): A finite set of infinitesimal generator $\{Q_1, Q_2,..., Q_r\}$ is said to be a basis for the Lie algebra L if $Q_i \in L$ and

1. $Q_1, Q_2,..., Q_r$ form a basis of the vector space L,
2. $[Q_i, Q_j] = c_{ijk} Q_k$.

3. The coefficients c_{ijk} $i, j, k = 1,2,...,r$, are called the structure constants of the Lie algebra.

Definition 2.4 (Andreas, 2009; Kolář, 1993): A Lie groups G is a smooth manifold and a group such that the multiplication $\mu : G \times G \to G$ is smooth. The inversion $\nu : G \to G$ is also smooth.

Theorem 2.5 (Second Fundamental Theorem of Lie) (Bluman and Kumei, 1989): Any two infinitesimal generators of an *r*-parameter Lie group, satisfy commutation relation of the form $[Q_i, Q_j] = c_{ijk} Q_k$, where $i, j, k = 1,2,\ldots, r$.

The commutator and the Jacobi identity, together with the capability to form real (or complex) linear combinations of the Q_i gives these infinitesimal generator the structure of the Lie algebra associated with the Lie group. Note that, in such a case the infinitesimal generators $Q_1, Q_2, \ldots, Q_r$ form a basis for a Lie algebra.

Theorem 2.6 (Ibragimov, 1999): For any variable x the function $F(x)$ is an invariant under the Lie group of transformation if and only if $XF(x) = 0$, where X is an infinitesimal generator.

Theorem 2.7 (Bluman and Kumei, 1989): The one-parameter Lie group $\bar{x} = F(x, \varepsilon)$ is equivalent to

$\bar{x} = e^{\varepsilon Q} x = x + \varepsilon Q x + \frac{\varepsilon^2}{2} Q^2 x + \frac{\varepsilon^3}{3!} Q^3 x + \ldots$, this is called Lie series. This theorem gives an approach to find a one-parameter group.

Lemma 2.8 (Muhsen and Maan, 2014a): The second order neutral delay differential equation, containing the infinitesimal generator ξ that obeys periodic property is given by $\xi(t, x) = \xi(t - \tau, x_\tau)$.

Lemma (2.8) implies that ξ does not depend on x. Now, let

$$x'' = f(t, x, x_\tau, x', x'_\tau, x''_\tau), \tag{1}$$

where $x = x(t), x' = x'(t), x'' = x''(t), x_\tau = x_\tau(t - \tau), x'_\tau = x'_\tau(t - \tau),$ and $x''_\tau\ x''_\tau(t - \tau).$

Equation (1) is a second order NDDE, the general infinitesimal generator of (1) is

$$X = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x} + \eta^{\tau} \frac{\partial}{\partial x_{\tau}} \tag{2}$$

where $\xi = \xi(t,x), \eta = \eta(t,x)$ and $\eta^{\tau} = \eta(t-\tau, x_{\tau})$. By Lemma 2.8, the determining equation for (1) is of the form

$$X^{(2)}\left(x''- f(t,x,x_{\tau},x',x'_{\tau},x''_{\tau})\right)\Big|_{(1)} = 0,$$

where

$$X^{(2)} = \xi \frac{\partial}{\partial t} + \eta^{x} \frac{\partial}{\partial x} + \eta^{x_{\tau}} \frac{\partial}{\partial x_{\tau}} + \eta^{x'} \frac{\partial}{\partial x'} + \eta^{x'_{\tau}} \frac{\partial}{\partial x'_{\tau}} + \eta^{x''} \frac{\partial}{\partial x''} + \eta^{x''_{\tau}} \frac{\partial}{\partial x''_{\tau}}, \tag{3}$$

and

$$\eta^{x}(t,x) = \eta(t,x),$$

$$\eta^{x_{\tau}}(t,x_{\tau}) = \eta(t-\tau,x_{\tau}),$$

$$\eta^{x'}(t,x,x') = \eta_1(t,x,x') = \eta_t(t,x) + [\eta_x(t,x) - \xi_t(t,x)]x' - \xi_x(t,x)(x')^2,$$

$$\eta^{x'_{\tau}}(t,x_{\tau},x'_{\tau}) = \eta_1(t-\tau,x_{\tau},x'_{\tau}) = \eta_t(t-\tau,x_{\tau}) + [\eta_x(t-\tau,x_{\tau}) - \xi_t(t-\tau,x_{\tau})]x'_{\tau} - \xi_x(t-\tau,x_{\tau})(x')^2,$$

$$\eta^{x''}(t,x,x',x'') = \eta_2(t,x,x',x'') = \eta_{tt}(t,x) + [2\eta_{tx}(t,x) - \xi_{tt}(t,x)]x' + [\eta_{xx}(t,x) - 2\xi_{tx}(t,x)](x')^2 - \xi_{xx}(t,x)(x')^3 + [\eta_x(t,x) - 2\xi_t(t,x)]x'' - 3\xi_x(t,x)x'x'',$$

$$\eta^{x''_{\tau}}(t,x_{\tau},x'_{\tau},x''_{\tau}) = \eta_2(t-\tau,x_{\tau},x'_{\tau},x''_{\tau}) = \eta_{tt}(t-\tau,x_{\tau}) + [2\eta_{tx}(t-\tau,x_{\tau}) - \xi_{tt}(t-\tau,x_{\tau})]x'_{\tau} + [\eta_{xx}(t-\tau,x_{\tau}) - 2\xi_{tx}(t-\tau,x_{\tau})](x'_{\tau})^2 - \xi_{xx}(t-\tau,x_{\tau})(x'_{\tau})^3 + [\eta_x(t-\tau,x_{\tau}) - 2\xi_t(t-\tau,x_{\tau})x''_{\tau} - 3\xi_x(t-\tau,x_{\tau})x'_{\tau}x''_{\tau}, \tag{4}$$

Algorithm 2.9 (Muhsen and Maan, 2014b) (Classification of second order non-linear RDDEs):

This algorithm is used to classify second order non-linear retarded delay differential equation to solvable Lie algebra.

i. Write the delay differential equation in the solved form.
ii. Write the general infinitesimal generator of the delay differential equation.

iii. Extend the infinitesimal generator acting on (x', x'_τ, x'').

iv. Apply the extended infinitesimal generator to the given delay differential equation to obtain invariance condition.

v. Substitute Equations (4) in the invariance condition.

vi. Split up invariance conditions by powers of the derivatives (x', x'_τ, x''), to give determining equations for the infinitesimal symmetry group.

vii. Then these determining equations are solved in the following steps

   a. Find the general solution of η and η^τ.
   b. Substitute these results in an equation that does not depend on the derivatives (x', x'_τ, x'') to obtain a polynomial of x.
   c. Solve the polynomial by comparing coefficient method.
   d. Find the solution of ξ. Then substitute the result to obtain the specific solution of η and η^τ.

viii. Substitute the infinitesimals ξ, η and η^τ in the general infinitesimal generator.

ix. Span Lie algebra of the given equation by the three infinitesimals generators corresponding to each $c_i, i = 1,2,..., n$ where c_i are arbitrary constants.

x. Compute the commutator table of the basis for Lie algebra.

xi. If the basis for Lie algebra satisfies the inclusion property, then the solvable Lie algebra is obtained.

3. Classification of Second Order Non-Linear NDDES to Solvable Lie Algebra

In this section a classification of non-linear homogenous and non-homogenous second order neutral delay differential equation to solvable Lie algebra is presented. We use Algorithm 2.9 to complete the classification of second order non-linear NDDEs with extension in steps,

iii. Extend the infinitesimal generator acting on second order neutral delay differential equation instead of retarded delay.

vi. Split up invariance conditions by powers of the derivatives $(x', x'_\tau, x'', x''_\tau)$, to give determining equations for the infinitesimal symmetry group.

vii. b. Substitute these results in an equation that does not depend on the derivatives $(x', x'_\tau, x'', x''_\tau)$ to obtain a polynomial of x.

To find the one-parameter group one need to add another step,

xii. Applied Theorem 2.7 on the results to get the one-parameter Lie group of the corresponding equation.

Example: Consider the second order non-linear homogenous NDDE

$$x''(t) + x''(t-\tau) + x'(t) + x'(t-\tau)x(t) = 0\,. \tag{5}$$

The general infinitesimal generator associated with Equation (5) is

$$X = \xi(t,x)\frac{\partial}{\partial t} + \eta(t,x)\frac{\partial}{\partial x} + \eta(t-\tau, x_\tau)\frac{\partial}{\partial x_\tau}. \tag{6}$$

The second order extension of (6) that is acting on neutral delay is

$$X^{(2)} = X + \eta_1(t,x,x')\frac{\partial}{\partial x'} + \eta_2(t,x,x',x'')\frac{\partial}{\partial x''} + \eta_1(t-\tau, x_\tau, x'_\tau)\frac{\partial}{\partial x'_\tau} +$$
$$\eta_2(t-\tau, x_\tau, x'_\tau, x''_\tau)\frac{\partial}{\partial x''_\tau}. \tag{7}$$

We get the invariance conditions by applying (7) to (5),

$$\eta_2 + \eta_2^\tau + \eta_1 + x'_\tau\,\eta + x\eta_1{}^\tau = 0\,,$$

where $\eta_1{}^\tau = \eta_1(t-\tau, x_\tau, x'_\tau)$, and $\eta_2^\tau = \eta_2(t-\tau, x_\tau, x'_\tau, x''_\tau)$.

Now, substituting the formula from (4), to obtain

$$\eta_{tt} + [2\eta_{tx} - \xi_{tt}]x' + [\eta_{xx} - 2\xi_{tx}](x')^2 - \xi_{xx}(x')^3 + [\eta_x - 2\xi_t]x'' - 3\xi_x x' x'' + \eta_{tt}^\tau +$$
$$[2\eta_{tx}^\tau - \xi_{tt}^\tau]x'_\tau + [\eta_{xx}^\tau - 2\xi_{tx}^\tau](x'_\tau)^2 - \xi_{xx}^\tau(x'_\tau)^3 + [\eta_x^\tau - 2\xi_t^\tau]x''_\tau - 3\xi_x^\tau x'_\tau x''_\tau + \eta_t +$$
$$[\eta_x - \xi_t]x' - \xi_x(x')^2 + x'_\tau\,\eta + x[\eta_t^\tau + [\eta_x^\tau - \xi_t^\tau]x'_\tau - \xi_x^\tau(x'_\tau)^2] = 0,$$

where $\eta^\tau = \eta(t-\tau, x_\tau)$, $\xi^\tau = \xi(t-\tau, x_\tau)$.

Equating the coefficients of the various monomials in the first, second orders of x and x_τ, we get the following determining equations (list in Table 1) for the symmetry group of Equation (5).

TABLE 1: The determining equations for the symmetry group of Equation (5)

MONOMIAL	COEFFCIENT	NUMBER OF EQUATION
1	$\eta_{tt}+\eta_{tt}^{\tau}+\eta_t+x\eta_t^{\tau}=0$	(a_1)
x'	$2\eta_{tx}-\xi_{tt}+\eta_x-\xi_t=0$	(a_2)
$(x')^2$	$\eta_{xx}-2\xi_{tx}-\xi_x=0$	(a_3)
$(x')^3$	$\xi_{xx}=0$	(a_4)
$x'x''$	$\xi_x=0$	(a_5)
x''	$\eta_x-2\xi_t=0$	(a_6)
x'_{τ}	$2\eta_{tx}^{\tau}-\xi_{tt}^{\tau}+\eta+x[\eta_x^{\tau}-\xi_t^{\tau}]=0$	(a_7)
$(x'_{\tau})^2$	$\eta_{xx}^{\tau}+2\xi_{tx}^{\tau}-x\xi_x^{\tau}=0$	(a_8)
$(x'_{\tau})^3$	$\xi_{xx}^{\tau}=0$	(a_9)
$x'_{\tau}x''_{\tau}$	$\xi_x^{\tau}=0$	(a_{10})
x''_{τ}	$\eta_x^{\tau}-2\xi_t^{\tau}=0$	(a_{11})

From (a_5), ξ does not depend on x. From (a_3), η is linear in x, so $\eta=g(t)x+h(t)$, where $g(t)$ and $h(t)$ are arbitrary functions of t. From (a_6), $\xi_t=\frac{1}{2}g$. From (a_{10}), ξ^{τ} is independent of x. From Lemma 2.8 $\xi=\xi^{\tau}$, so $\xi_t=\xi_t^{\tau}$ and $\xi_t^{\tau}=\frac{1}{2}g$. From ($a_{11}$), $\eta_x^{\tau}=g$. This implies that $\eta^{\tau}=g(t)x+k(t-\tau)$, where $k(t-\tau)$ is an arbitrary function. If $\tau=0$, then $k=h$.

Now, from (a_1), $g_{tt}x+h_{tt}+g_{tt}x+k_{tt}+g_tx+h_t+g_tx^2+k_tx=0$. Equating the coefficients of the various terms, we obtain

$$g_t=0, \tag{8}$$

$$2g_{tt}+g_t+k_t=0, \tag{9}$$

$$h_{tt}+k_{tt}+h_t=0, \tag{10}$$

which means that $g(t)$, $h(t)$ and $k(t-\tau)$ are the solutions of (5).

From (8), $g_t=0$, then $g=c_1$. Since $g=2\xi_t$, this implies that $\xi=c_2t+c_3$, where $c_2=\frac{1}{2}c_1$. From (9), $k_t=0$, so $k=c_4$. From (10), $h_t=-h_{tt}$. This implies that $h = -h_t + c_5$. From above $\eta = c_1 x - h_t + c_5$, and $\eta^{\tau} = c_1 x + c_4$, where c_i, $i = 1, 2, ..., 5$ are arbitrary constants.

Recall from Equation (6) that the infinitesimal generator of Equation (5) is

$$X = \xi(t,x)\frac{\partial}{\partial t} + \eta(t,x)\frac{\partial}{\partial x} + \eta(t-\tau, x_{\tau})\frac{\partial}{\partial x_{\tau}}.$$

Let $x_{\tau} = u$, then

$$X = (c_2 t + c_3)\frac{\partial}{\partial t} + (c_1 x - h_t + c_5)\frac{\partial}{\partial x} + (c_1 x + c_4)\frac{\partial}{\partial u},$$

Thus, the Lie algebra of Equation (5) is spanned by the following three infinitesimal generators corresponding to each c_i.

$$Q_1 = x(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}),\ Q_2 = t\frac{\partial}{\partial t},\ Q_3 = \frac{\partial}{\partial t},\ Q_4 = \frac{\partial}{\partial u},\ Q_5 = \frac{\partial}{\partial x},$$

with infinite dimensional Lie subalgebra $Q_6 = -h_t \frac{\partial}{\partial x}$. The commutator table is given in Table 2.

Thus, the algebra $L_5 = \{Q_1, Q_2, Q_3, Q_4, Q_5\}$, spanned by Q_1, Q_2, Q_3, Q_4, Q_5, is Lie algebra of Equation (5). The subspaces $L_1 = \{Q_1\}$, $L_2 = \{Q_1, Q_2\}$, $L_3 = \{Q_1, Q_2, Q_3\}$, $L_4 = \{Q_1, Q_2, Q_3, Q_4\}$, are Lie subalgebras of L_5 of dimensions one, two, three, four and five respectively. Furthermore these Lie subalgebras satisfies the inclusion property: $L_1 \subset L_2 \subset L_3 \subset L_4 \subset L_5$, hence, by Definition 2.1, L_5 is a solvable Lie algebra of Equation (5).

TABLE 2: The commutator table for the generators of the symmetry group of Equation (5)

$[Q_i, Q_j]$	Q_1	Q_2	Q_3	Q_4	Q_5
Q_1	0	0	0	0	$-Q_4 - Q_5$
Q_2	0	0	Q_3	0	0
Q_3	0	$-Q_3$	0	0	0
Q_4	0	0	0	0	0
Q_5	$Q_4 + Q_5$	0	0	0	0

Now, by applying Theorem 2.7 on $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6$ one can get the one-parameter Lie groups generator by these space.

$$Q_1: \quad \bar{t} = \sum_{n=0}^{\infty} \varepsilon_1^n t, \quad \bar{x} = \sum_{n=1}^{\infty} n\varepsilon_1^{n-1} x, \quad \bar{u} = \varepsilon_1 x + \sum_{n=0}^{\infty} \varepsilon_1^n u.$$

$$Q_2: \quad \bar{t} = \sum_{n=1}^{\infty} n\varepsilon_2^{n-1} t, \quad \bar{x} = \sum_{n=0}^{\infty} \varepsilon_2^n x, \quad \bar{u} = \sum_{n=0}^{\infty} \varepsilon_2^n u.$$

$$Q_3: \quad \bar{t} = t + \varepsilon_3, \quad \bar{x} = x, \quad \bar{u} = u.$$

$$Q_4: \quad \bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = u + \varepsilon_4.$$

$$Q_5: \quad \bar{t} = t, \quad \bar{x} = x + \varepsilon_5, \quad \bar{u} = u.$$

$$Q_6: \quad \bar{t} = t, \quad \bar{x} = x - \varepsilon_6 h_t, \quad \bar{u} = u.$$

Here $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6$ are the parameters of the one-parameter groups generated by $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6$, respectively.

Remark: Suppose Equation (5) is non-homogenous, i.e.

$$x''(t) + x''(t-\tau) + x'(t) + x'(t-\tau)x(t) = r(t), \tag{11}$$

where $r(t)$ is arbitrary non-integrable function. Then, the Lie algebra of Equation (11) is spanned by

$$Q_1 = x(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}), \; Q_2 = t\frac{\partial}{\partial t}, \; Q_3 = \frac{\partial}{\partial t}, \; Q_4 = \frac{\partial}{\partial u}, \; Q_5 = h\frac{\partial}{\partial x},$$

where Q_5 is infinite dimensional Lie subalgebra, and the commutator table is shown in Table 3.

Then, the algebra $L_4 = \{Q_1, Q_2, Q_3, Q_4\}$ spanned by Q_1, Q_2, Q_3, Q_4, is solvable Lie algebra of Equation (11).

TABLE 3: The commutator table for the generators of the symmetry group of Equation (11)

$[Q_i, Q_j]$	Q_1	Q_2	Q_3	Q_4
Q_1	0	0	0	0
Q_2	0	0	Q_3	0
Q_3	0	$-Q_3$	0	0
Q_4	0	0	0	0

Applied Theorem 2.7 to these space to get

$$Q_1: \quad \bar{t} = \sum_{n=0}^{\infty} \varepsilon_1^n t, \quad \bar{x} = \sum_{n=1}^{\infty} n\varepsilon_1^{n-1} x, \quad \bar{u} = \varepsilon_1 x + \sum_{n=0}^{\infty} \varepsilon_1^n u.$$

$$Q_2: \quad \bar{t} = \sum_{n=1}^{\infty} n\varepsilon_2^{n-1} t, \quad \bar{x} = \sum_{n=0}^{\infty} \varepsilon_2^n x, \quad \bar{u} = \sum_{n=0}^{\infty} \varepsilon_2^n u.$$

$$Q_3: \quad \bar{t} = t + \varepsilon_3, \quad \bar{x} = x, \quad \bar{u} = u.$$

$$Q_4: \quad \bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = u + \varepsilon_4.$$

$$Q_5: \quad \bar{t} = t, \quad \bar{x} = x + \varepsilon_5 h, \quad \bar{u} = u.$$

Where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5$ are the parameters of the one-parameter groups generated by L_4 with Q_5 respectively.

4. Conclusion

This paper extends the classification of second order non-linear RDDEs to the classification of second order non-linear NDDEs as solvable Lie algebras. Then, the one-parameter Lie group are obtained by Lie series corresponding to NDDEs, which can be used for general analysis of the equations. These results and the successful implementation form the basis for the classification of non-linear delay differential equations of neutral type to solvable Lie algebra. Thus the classification of second order non-linear DDEs to solvable Lie algebra is completed. Results of this paper could be extended to higher order non-linear delay differential equations.

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References

Andreas, Ĉ. (2009). *Lie algebras and representation theory*. Universität Wien: Nordbergstr.

Batzel, J. J. and H. T. Tran, H. T. (2000). Stability of the human respiratory control system I. Analysis of a two-dimensional delay state-space model. *J Math Biol*. **41**: 45–79.

Bellen A., and Zennaro, M. (2003). *Numerical methods for delay differential equations. Numerical mathematics and scientific computation*. New York, Clarendon Press, Oxford University Press.

Bluman, G. W. and Kumei, S. (1989). *Symmetries and differential equations*. New York: Sprinder.

Gorecki, H., Fuksa, S. Grabowski, P. and Korytowski, A. (1989). *Analysis and synthesis of time delay systems*. New York: John Wiley and Sons.

Hill, J. M. (1982). *Solution of differential equation by means of one-parameter groups*. London.

Humi, M. and Miller, W. (1988). *Second course in ordinary differential equations for scientists and engineers.* New York: Springer.

Ibragimov, N. H. (1999). *Elementary Lie group analysis and ordinary differential equations*. London: Wiley.

Kolář, I. and Slovák, J. (1993). *Natural operations in differential geometry". Department of Algebra and Geometry*. Faculty of science. Masaryk University. Berlin:Springer.

Muhsen, L. and Maan, N. (2014a). New approach to classify second order linear delay diferential equations with constant coeficients to solvable Lie algebra. *International journal of mathematical analysis*. **20**: 973-993.

Muhsen, L. and Maan, N. (2014b). Classification of second order non-linear retarded delay differential equations to solvable Lie algebra. *Proceeding in Simposium Kebangsaan Sains Matematik ke-22*. Shah Alam: Malaysia.

Nagy, T. K.,Stépán, G. and Moon, F. C. (2001). Subcritical Hopf bifurcation in the delay equation model for machine tool vibrations, *Nonlinear Dyn*. **26**: 121–142.

Oliveri, F. (2010). Lie symmetries of differential equations: classical results and recent contributions, *Symmetry Journal*. **2**: 658-706.

Olver, P. J. (1993). *Application of Lie groups to differential equations*. 2nd ed. New York. Springer, Verlag.

Pue-on, P. (2009). Group classification Of second-order delay differential equations. *Commun Nonlinear Sci Numer Simul*. **15**: 1444-1453.

Thanthanuch, J. and Meleshko, S. V. (2004). On definition of an admitted Lie group for functional differential equations. *Commun Nonlinear Sci Numer Simul*. **9**: 117-125.